

A note on the partition dimension of Cartesian product graphs

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Abstract

Let $G = (V, E)$ be a connected graph. The distance between two vertices $u, v \in V$, denoted by $d(u, v)$, is the length of a shortest $u - v$ path in G . The distance between a vertex $v \in V$ and a subset $P \subset V$ is defined as $\min\{d(v, x) : x \in P\}$, and it is denoted by $d(v, P)$. An ordered partition $\{P_1, P_2, \dots, P_t\}$ of vertices of a graph G , is a *resolving partition* of G , if all the distance vectors $(d(v, P_1), d(v, P_2), \dots, d(v, P_t))$ are different. The *partition dimension* of G , denoted by $pd(G)$, is the minimum number of sets in any resolving partition of G . In this article we study the partition dimension of Cartesian product graphs. More precisely, we show that for all pairs of connected graphs G, H , $pd(G \times H) \leq pd(G) + pd(H)$ and $pd(G \times H) \leq pd(G) + \dim(H)$, where $\dim(H)$ denotes the metric dimension of H . Consequently, we show that $pd(G \times H) \leq \dim(G) + \dim(H) + 1$.

Keywords: Resolving sets, resolving partition, partition dimension, Cartesian product.

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1 Introduction

The concepts of resolvability and location in graphs were described independently by Harary and Melter [9] and Slater [17], to define the same structure in a graph. After these papers were published several authors developed diverse theoretical works about this topic [2, 3, 4, 5, 6, 7, 8, 14]. Slater described the usefulness of these ideas into long range aids to navigation [17]. Also, these concepts have some applications in chemistry for representing chemical compounds [12, 13] or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [15]. Other applications of this concept to navigation of robots in networks and other areas appear in [5, 11, 14]. Some variations on resolvability or location have been appearing in the literature, like those about conditional resolvability [16], locating domination [10], resolving domination [1] and resolving partitions [4, 7, 8].

Given a graph $G = (V, E)$ and an ordered set of vertices $S = \{v_1, v_2, \dots, v_k\}$ of G , the *metric representation* of a vertex $v \in V$ with respect to S is the vector $r(v|S) = (d(v, v_1), d(v, v_2), \dots, d(v, v_k))$, where $d(v, v_i)$, with $1 \leq i \leq k$, denotes the distance between the vertices v and v_i . We say that S is a *resolving set* of G if for every pair of distinct vertices $u, v \in V$, $r(u|S) \neq r(v|S)$. The *metric dimension*¹ of G is the minimum cardinality of any resolving set of G , and it is denoted by $\dim(G)$. The metric dimension of graphs is studied in [2, 3, 4, 5, 6, 18].

Given an ordered partition $\Pi = \{P_1, P_2, \dots, P_t\}$ of the vertices of G , the *partition representation* of a vertex $v \in V$ with respect to the partition Π is the vector $r(v|\Pi) = (d(v, P_1), d(v, P_2), \dots, d(v, P_t))$, where $d(v, P_i)$, with $1 \leq i \leq t$, represents the distance between the vertex v and the set P_i , that is $d(v, P_i) = \min_{u \in P_i} \{d(v, u)\}$. We say that Π is a *resolving partition* of G if for every pair of distinct vertices $u, v \in V$, $r(u|\Pi) \neq r(v|\Pi)$. The *partition dimension* of G is the minimum number of sets in any resolving partition of G and it is denoted by $pd(G)$. The partition dimension of graphs is studied in [4, 7, 8, 18]. It is natural to think that the partition dimension and metric dimension are related; in [7] it was shown that for any nontrivial connected graph G we have

$$pd(G) \leq \dim(G) + 1. \quad (1)$$

The study of relationships between invariants of Cartesian product graphs

¹Also called locating number.

and invariants of its factors appears frequently in research about graph theory. In the case of resolvability, the relationships between the metric dimension of the Cartesian product graphs and the metric dimension of its factors was studied in [2, 3]. An open problem on the dimension of Cartesian product graphs is to prove (or finding a counterexample) that for all pairs of graphs G, H ; $\dim(G \times H) \leq \dim(G) + \dim(H)$. In the present paper we study the case of resolving partition in Cartesian product graphs, by giving some relationships between the partition dimension of Cartesian product graphs and the partition dimension of its factors. More precisely, we show that for all pairs of connected graphs G, H ; $pd(G \times H) \leq pd(G) + pd(H)$ and $pd(G \times H) \leq pd(G) + \dim(H)$. Consequently, we show that $pd(G \times H) \leq \dim(G) + \dim(H) + 1$.

We recall that the Cartesian product of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \times G_2 = (V, E)$, such that $V = \{(a, b) : a \in V_1, b \in V_2\}$ and two vertices $(a, b) \in V$ and $(c, d) \in V$ are adjacent in $G_1 \times G_2$ if and only if, either $a = c$ and $bd \in E_2$ or $b = d$ and $ac \in E_1$.

The following well known fact will be used several times.

Remark 1. Let the graph $G_i = (V_i, E_i)$ and let $S_i \subset V_i$, $i \in \{1, 2\}$. For every $(a, b) \in V_1 \times V_2$, it follows $d_{G_1 \times G_2}((a, b), S_1 \times S_2) = d_{G_1}(a, S_1) + d_{G_2}(b, S_2)$.

2 The partition dimension of Cartesian product graphs

Theorem 2. For any connected graphs G_1 and G_2 ,

$$pd(G_1 \times G_2) \leq pd(G_1) + pd(G_2).$$

Proof. Let $\Pi_1 = \{A_1, A_2, \dots, A_k\}$ and $\Pi_2 = \{B_1, B_2, \dots, B_t\}$ be resolving partitions of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ respectively. Let us show that $\Pi = \{A_1 \times B_1, A_1 \times B_2, \dots, A_1 \times B_t, A_2 \times B_1, A_3 \times B_1, \dots, A_k \times B_1, C\}$, with $C = (V_1 \times V_2) - ((V_1 \times B_1) \cup (A_1 \times V_2))$ is a resolving partition of $G_1 \times G_2$.

Let $(a, b), (c, d)$ be two different vertices of $V_1 \times V_2$. If $a = c$, then there exists $B_i \in \Pi_2$ such that $d_{G_2}(b, B_i) \neq d_{G_2}(d, B_i)$. Hence we have

$$\begin{aligned} d_{G_1 \times G_2}((a, b), A_1 \times B_i) &= d_{G_1}(a, A_1) + d_{G_2}(b, B_i) \\ &\neq d_{G_1}(c, A_1) + d_{G_2}(d, B_i) \\ &= d_{G_1 \times G_2}((c, d), A_1 \times B_i) \end{aligned}$$

Now, if $a \neq c$ then we have the following cases:

Case 1: Let $a \in A_i$ and $c \in A_j$, with $i \neq j$. If we suppose,

$$d_{G_1 \times G_2}((a, b), A_i \times B_1) = d_{G_1 \times G_2}((c, d), A_i \times B_1)$$

and

$$d_{G_1 \times G_2}((a, b), A_j \times B_1) = d_{G_1 \times G_2}((c, d), A_j \times B_1),$$

we obtain

$$\begin{aligned} d_{G_2}(b, B_1) &= d_{G_1 \times G_2}((a, b), A_i \times B_1) \\ &= d_{G_1 \times G_2}((c, d), A_i \times B_1) \\ &= d_{G_1}(c, A_i) + d_{G_2}(d, B_1) \\ &= d_{G_1}(c, A_i) + d_{G_1 \times G_2}((c, d), A_j \times B_1) \\ &= d_{G_1}(c, A_i) + d_{G_1 \times G_2}((a, b), A_j \times B_1) \\ &= d_{G_1}(c, A_i) + d_{G_1}(a, A_j) + d_{G_2}(b, B_1), \end{aligned}$$

a contradiction.

Case 2: If $a, c \in A_i$ then we have the following subcases.

Case 2.1: $b, d \in B_l$. Let $A_j \in \Pi_1$, such that $d_{G_1}(a, A_j) \neq d_{G_1}(c, A_j)$. In this case, if $d_{G_2}(b, B_1) = d_{G_2}(d, B_1)$ then we have

$$\begin{aligned} d_{G_1 \times G_2}((a, b), A_j \times B_1) &= d_{G_1}(a, A_j) + d_{G_2}(b, B_1) \\ &\neq d_{G_1}(c, A_j) + d_{G_2}(d, B_1) \\ &= d_{G_1 \times G_2}((c, d), A_j \times B_1). \end{aligned}$$

On the contrary, if $d_{G_2}(b, B_1) \neq d_{G_2}(d, B_1)$ then we have

$$\begin{aligned} d_{G_1 \times G_2}((a, b), A_i \times B_1) &= d_{G_2}(b, B_1) \\ &\neq d_{G_2}(d, B_1) \\ &= d_{G_1 \times G_2}((c, d), A_i \times B_1). \end{aligned}$$

Case 2.2: $b \in B_j$ and $d \in B_l$, $j \neq l$. This case is analogous to Case 1.

Therefore for every different vertices $(a, b), (c, d) \in V_1 \times V_2$, we have $r((a, b)|\Pi) \neq r((c, d)|\Pi)$. \square

By (1) we obtain the following direct consequence of Theorem 2.

Corollary 3. *For any connected graphs G_1 and G_2 ,*

$$pd(G_1 \times G_2) \leq pd(G_1) + dim(G_2) + 1.$$

As we can see below, the above relationship can be improved.

Theorem 4. *For any connected graphs G_1 and G_2 ,*

$$pd(G_1 \times G_2) \leq pd(G_1) + dim(G_2).$$

Proof. Let $\Pi = \{A_1, A_2, \dots, A_k\}$ be a resolving partition of $G_1 = (V_1, E_1)$, let $S = \{u_1, u_2, \dots, u_t\}$ be a resolving set of $G_2 = (V_2, E_2)$ and let $C = V_1 \times V_2 - ((V_1 \times \{u_1\}) \cup (A_1 \times \{u_2\}) \cup \dots \cup (A_1 \times \{u_t\}))$. Let us show that $\Pi_1 = \{A_1 \times \{u_1\}, A_2 \times \{u_1\}, \dots, A_k \times \{u_1\}, A_1 \times \{u_2\}, A_1 \times \{u_3\}, \dots, A_1 \times \{u_t\}, C\}$ is a resolving partition of $G_1 \times G_2$.

Let $(a, b), (c, d)$ be two different vertices of $V_1 \times V_2$. If $a = c$, then $b \neq d$. Thus, there exist $u_j \in S$ such that $d_{G_2}(b, u_j) \neq d_{G_2}(d, u_j)$. Hence,

$$\begin{aligned} d_{G_1 \times G_2}((a, b), A_1 \times \{u_j\}) &= d_{G_1}(a, A_1) + d_{G_2}(b, u_j) \\ &\neq d_{G_1}(c, A_1) + d_{G_2}(d, u_j) \\ &= d_{G_1 \times G_2}((c, d), A_1 \times \{u_j\}) \end{aligned}$$

Now, if $a \neq c$ we have two cases:

Case 1: $a \in A_i$ and $c \in A_j, j \neq i$. Let us suppose, $d_{G_2}(b, u_1) \leq d_{G_2}(d, u_1)$. In this case we have

$$\begin{aligned} d_{G_1 \times G_2}((a, b), A_i \times \{u_1\}) &= d_{G_2}(b, u_1) \\ &\leq d_{G_2}(d, u_1) \\ &< d_{G_1}(c, A_i) + d_{G_2}(d, u_1) \\ &= d_{G_1 \times G_2}((c, d), A_i \times \{u_1\}). \end{aligned}$$

Analogously, if $d_{G_2}(b, u_1) \geq d_{G_2}(d, u_1)$ we obtain

$$d_{G_1 \times G_2}((a, b), A_j \times \{u_1\}) > d_{G_1 \times G_2}((c, d), A_j \times \{u_1\}).$$

Case 2: $a, c \in A_i$. Let us suppose $d_{G_2}(b, u_1) = d_{G_2}(d, u_1)$. Since there exists $j \neq i$, such that $d_G(a, A_j) \neq d_G(c, A_j)$, we have

$$\begin{aligned} d_{G_1 \times G_2}((a, b), A_j \times \{u_1\}) &= d_{G_1}(a, A_j) + d_{G_2}(b, u_1) \\ &\neq d_{G_1}(c, A_j) + d_{G_2}(d, u_1) \\ &= d_{G_1 \times G_2}((c, d), A_j \times \{u_1\}). \end{aligned}$$

If $d_{G_2}(b, u_1) \neq d_{G_2}(d, u_1)$, we have $d_{G_1 \times G_2}((a, b), A_i \times \{u_1\}) = d_{G_2}(b, u_1) \neq d_{G_2}(d, u_1) = d_{G_1 \times G_2}((c, d), A_i \times \{u_1\})$. Therefore, for every different vertices $(a, b), (c, d)$ we have $r((a, b)|\Pi_1) \neq r((c, d)|\Pi_1)$. \square

In order to give some examples we emphasize the following well known values for the metric dimension of the complete graph, K_n , the path graph, P_n , the cycle graph, C_n , and the star graph, $K_{1,n}$.

Remark 5.

- (i) $\dim(K_n) = n - 1$ ($n \geq 2$).
- (ii) $\dim(P_n) = 1$.
- (iii) $\dim(C_n) = 2$.
- (iv) $\dim(K_{1,n}) = n - 1$ ($n \geq 2$).

We note that there are graphs for which Theorem 2 estimates $pd(G_1 \times G_2)$ better than Theorem 4 and vice versa. For example Theorem 2 leads to $pd(K_n \times P_n) \leq n + 2$ while Theorem 4 gives $pd(K_n \times P_n) \leq n + 1$. On the contrary, if G denotes the unicyclic graph described below, Theorem 2 leads to $pd(G \times G) \leq 12$ while Theorem 4 gives $pd(G \times G) \leq 15$. In the above example the unicyclic graph G is composed by fifteen vertices, where the set $1, 2, 3$ form a triangle and the remaining twelve vertices are leaves: the leaves $4, 5, 6$ and 7 are adjacent to 1 , the leaves $8, 9, 10$ and 11 are adjacent to 2 , and the leaves $12, 13, 14$ and 15 are adjacent to 3 . In this case $\Pi = \{\{4, 1, 2, 3\}, \{8\}, \{12\}, \{5, 9, 13\}, \{6, 10, 14\}, \{7, 11, 15\}\}$ is a resolving partition and $S = \{4, 5, 6, 8, 9, 10, 12, 13, 14\}$ is a resolving set.

As a direct consequence of above theorem and (1) we deduce the following interesting result.

Corollary 6. *For any connected graphs G_1 and G_2 ,*

$$pd(G_1 \times G_2) \leq \dim(G_1) + \dim(G_2) + 1.$$

One example of graphs for which the equality holds in Corollary 6 (and also in Corollary 7 (ii)) are the graphs belonging to the family of grid graphs: $pd(P_r \times P_t) = 3$.

By Remark 5 we obtain the following particular cases of Theorem 4.

Corollary 7. *For any connected graph G ,*

- (i) $pd(G \times K_n) \leq pd(G) + n - 1$.
- (ii) $pd(G \times P_n) \leq pd(G) + 1$.
- (iii) $pd(G \times C_n) \leq pd(G) + 2$.
- (iv) $pd(G \times K_{1,n}) \leq pd(G) + n - 1$.

3 Open problems

1. To prove (or finding a counterexample) that for all pairs of graphs G, H ;
 $\dim(G \times H) \leq \dim(G) + \dim(H)$.
2. To provide lower bounds for $pd(G \times H)$.

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